AN EXTENSION OF LIE'S THEOREM ON ISOTHERMAL FAMILIES

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1. *Conformal Representation.*—Let a surface Σ be represented conformally upon a plane π with cartesian coordinates (x, y). The linear-element of Σ is then of the form

$$ds^2 = E(x, y)(dx^2 + dy^2), \tag{1}$$

where $E(x, y) > 0$. The parametric curves x = const. and y = const. form an isothermal net on Σ.

It is well known that $h(x, y) = c$, where h is a harmonic function of (x, y), that is, h satisfies the Laplace equation, $h_{xx} + h_{yy} = 0$, defines an isothermal family of curves on Σ. The constant c is called the isothermal parameter.

The converse of the preceding statement is not valid. That is, if $g(x, y)$ = const. defines an isothermal family of curves on Σ with linear-element of the form (1), then it does not follow necessarily that g is a harmonic function, but it must be a function of a harmonic function. The exact statement which is due to Lie is as follows. The family of curves $g(x, y)$ = const. represents an isothermal system on the surface Σ with linear-element of the form (1), if and only if g satisfies the partial differential equation of third order

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \text{arc tan} \frac{g_y}{g_x} = 0. \tag{2}$$

This means geometrically that the angle θ between the ∞^1 curves $g(x, y)$ = const. and the isothermal family x = const. (or y = const.) is a harmonic function of (x, y).

2. *Statement of Our Problem.*—We propose to give the necessary and sufficient condition that $g(x, y)$ = const., represent an isothermal family upon a surface Σ when (x, y) are general curvilinear coordinates on Σ. That is, when the linear-element of Σ is of the form

$$ds^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2, \tag{3}$$

where $H^2 = EG - F^2 > 0$, we shall determine the necessary and sufficient condition that $g(x, y)$ = const. represents an isothermal system of curves on Σ. This leads to a wide extension of the theorem of Lie, which is stated above.

The preceding condition is simpler when the parametric curves form an orthogonal net, that is, when $F = 0$. Of course, the simplest condition is (2) when the parametric curves form an isothermal net and (x, y) are isothermal coordinates.

As an application of our preceding work, we shall obtain the necessary and sufficient condition that the ∞^1 curves $g(x, y) = \text{const.}$ shall represent an isothermal family upon the Monge surface $\Sigma : z = f(x, y)$.

Finally the condition is found that the level curves $z = \text{const.}$ of the surface $\Sigma : z = f(x, y)$ be an isothermal family. This is applied to the mapping upon a plane π of the loxodromes (relative to the level curves) of the surface Σ, showing that they can be represented by straight lines for a sphere (Mercator) and a spheroid (Lambert), (and for any minimal surface and also for any surface of revolution with axis perpendicular to the xy-plane), but not for an ellipsoid of three unequal axes. Here use is made of a theorem of Kasner, which states that the complete system of ∞^2 isogonal trajectories of a given family of curves is linear if and only if the given family is isothermal.

3. *The Condition When the Minimal Lines Are Given in the Finite Form.*—Let (u, v) denote the minimal coordinates of any point on the surface Σ. Then $u = x + iy$, $v = x - iy$ where (x, y) are the isothermal coordinates defined in (1).

If a general point transformation is applied to the plane π upon which the surface Σ is conformally represented by means of the equation (1), it is found that u and v are given by the general expressions

$$u = \phi(x, y), \qquad v = \psi(x, y), \tag{4}$$

where ϕ and ψ are conjugate complex functions of the real variables (x,y). Of course, (x, y) are now general curvilinear coordinates of any point on Σ since the linear-element of Σ is of the form (3). The finite forms of the equations of the minimal lines are $\phi(x, y) = \text{const.}$ and $\psi(x, y) = \text{const.}$

We seek the condition for an isothermal family in the general curvilinear coordinates (x, y). According to Lie's theorem, any isothermal family is defined in minimal coordinates (u, v) by a differential equation of the form

$$\log \frac{dv}{du} = \lambda(u) + \mu(v). \tag{5}$$

By substituting (4) into this equation, we find that the ∞^1 curves defined by the differential equation of first order $dy/dx = p = p(x, y)$ form an isothermal family if and only if the function p of (x, y) satisfies an equation of the form

$$\log \frac{\psi_x + p\psi_y}{\phi_x + p\phi_y} = \lambda(\phi) + \mu(\psi). \tag{6}$$

The functions λ of ϕ and μ of ψ must be eliminated by partial differentiation. Firstly upon applying the operation $\phi_y\partial/\partial x - \phi_x\partial/\partial y$ and simplifying, and secondly the operation $\psi_y\partial/\partial x - \psi_x\partial/\partial y$ to the above equation, we obtain the expression

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)\left[\frac{(\phi_y(\partial/\partial x) - \phi_x(\partial/\partial y))\log \dfrac{\psi_x + p\psi_y}{\phi_x + p\phi_y}}{\phi_x\psi_y - \phi_y\psi_x}\right] = 0. \quad (7)$$

This condition is the necessary and sufficient condition that the curves defined by the differential equation $dy/dx = p(x, y)$ form an isothermal family on the surface Σ whose minimal curves are given in the finite form by $\phi(x, y) =$ const. and $\psi(x, y) =$ const.

4. *The Condition* (7) *in Terms of the Differential Equations of the Minimal Curves.*—For this purpose, let us define $\alpha(x, y)$ and $\beta(x, y)$ by the equations

$$\phi_x = -\alpha\phi_y, \quad \psi_x = -\beta\psi_y,$$

so that the differential equations of the minimal curves are $dy/dx = \alpha$ and $dy/dx = \beta$.

Upon substituting these into (7) and simplifying, we obtain the expression ultimately

$$\left[\frac{\partial^2}{\partial x^2} + (\alpha + \beta)\frac{\partial^2}{\partial x\partial y} + \alpha\beta\frac{\partial^2}{\partial y^2}\right]\log\frac{p - \beta}{p - \alpha} + \left(\frac{\beta_x + \alpha\beta_y}{\alpha - \beta}\right)\times$$
$$\left(\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}\right)\log\frac{p - \beta}{p - \alpha} - \left(\frac{\alpha_x + \beta\alpha_y}{\alpha - \beta}\right)\left(\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)\log\frac{p - \beta}{p - \alpha} +$$
$$(\beta\alpha_{yy} - \alpha\beta_{yy} + \alpha_{xy} - \beta_{xy}) + \left(\frac{\alpha_y - \beta_y}{\alpha - \beta}\right)(\alpha\beta_y - \beta\alpha_y + \beta_x - \alpha_x) = 0. \quad (8)$$

This is the necessary and sufficient condition that the family of curves defined by the differential equation $dy/dx = p = p(x, y)$ is an isothermal family when the minimal curves of the surface Σ are given by the differential equations $dy/dx = \alpha(x, y)$ and $dy/dx = \beta(x, y)$.

5. *The Condition* (8) *in Terms of the General Form* (3) *of the Linear-Element of the Surface* Σ.—If (x, y) are curvilinear coordinates on the surface Σ such that the linear-element of Σ is given by the general equation (3), it follows that α and β must be given by

$$\alpha = \frac{-1}{G}(F - iH), \quad \beta = -\frac{1}{G}(F + iH). \quad (9)$$

Let θ denote the expression

$$\theta = \text{arc cot } \frac{1}{H}(pG + F). \tag{10}$$

This is actually the angle θ between any curve of the family $dy/dx = p(x, y)$ and the parametric curves $x = \text{const.}$

Substituting the values of α and β as given by equations (9), we find that the equation (8) assumes the form

$$H\left[G\frac{\partial^2\theta}{\partial x^2} - 2F\frac{\partial^2\theta}{\partial x\partial y} + E\frac{\partial^2\theta}{\partial y^2}\right] + [HG_x - GH_x + FH_y - HF_y]\frac{\partial\theta}{\partial x} +$$

$$[FH_x - HF_x + \frac{1}{G}\{-EHG_y + 2FHF_y + (EG - 2F^2)H_y\}]\frac{\partial\theta}{\partial y} +$$

$$\frac{H}{G}[HF_{yy} - FH_{yy} + GH_{xy} - HG_{xy}] + \frac{1}{G^2}(GH_y + HG_y)(FH_y - HF_y) +$$

$$\frac{1}{G^2}(H^2G_xG_y - G^2H_xH_y) = 0. \tag{11}$$

This is our extension of Lie's theorem on isothermal familes. If it is desirable to write the above equation in a form which does not contain partial derivatives of H, we find the form

$$2H^2\left[G\frac{\partial^2\theta}{\partial x^2} - 2F\frac{\partial^2\theta}{\partial x\partial y} + E\frac{\partial^2\theta}{\partial y^2}\right] + [(EG - 2F^2)G_x - G^2E_x + 2FGF_x +$$

$$EFG_y + GFE_y - 2EGF_y]\frac{\partial\theta}{\partial x} + [EFG_x + GFE_x - 2EGF_x - E^2G_y +$$

$$(EG - 2F^2)E_y + 2EFF_y]\frac{\partial\theta}{\partial y} + H[-FE_{yy} - \frac{EF}{G}G_{yy} + 2EF_{yy} + GE_{xy} -$$

$$\frac{1}{G}(EG - 2F^2)G_{xy} - 2FF_{xy}]$$

$$+\frac{1}{H}\left[\begin{array}{l} GFE_y^2 + \frac{EF}{G^2}(2EG - F^2)G_y^2 + 4EFF_y^2 + \frac{F}{G}(EG + F^2)\times \\ E_yG_y - (EG + 3F^2)E_yF_y - \frac{E}{G}(3EG + F^2)G_yF_y - G^2E_xE_y - \\ F^2E_xG_y + 2FGE_xF_y + \frac{1}{G^2}(2H^4 - E^2G^2)G_xG_y - F^2E_yG_x + \\ 2EFG_xF_y + 2EFG_yF_x + 2GFE_yF_x - 2(EG + F^2)F_xF_y \end{array}\right]$$

$$= 0. \tag{12}$$

6. The Condition (12) When the Parametric Curves Are Orthogonal.—The parametric curves are orthogonal if and only if $F = 0$. In that event the condition (12) becomes

$$2EG\left(G\frac{\partial^2\theta}{\partial x^2} + E\frac{\partial^2\theta}{\partial y^2}\right) + G(EG_x - GE_x)\frac{\partial\theta}{\partial x} + E(GE_y - EG_y)\frac{\partial\theta}{\partial y} +$$

$$(EG)^{1/2}(GE_{xy} - EG_{xy}) + (EG)^{-1/2}(E^2G_xG_y - G^2E_xE_y) = 0, \quad (13)$$

where $\theta = \text{arc cot } p(G/E)^{1/2}$.

This is the necessary and sufficient condition that the ∞^1 curves defined by the differential equation $dy/dx = p(x, y)$ be an isothermal family when the parametric curves on the surface Σ form an orthogonal net.

Of course, if the parametric curves form an isothermal net and if x and y are isothermal parameters, then (13) reduces to Lie's theorem stating that θ is a harmonic function.

7. The Condition (12) When the Surface Σ Is Given by the Monge Equation $z = f(x, y)$.—In this case $E = 1 + f_x^2$, $F = f_xf_y$, $G = 1 + f_y^2$. Substituting these values into (12), we obtain the expression

$$(1 + f_x^2 + f_y^2)\left[(1 + f_y^2)\frac{\partial^2\theta}{\partial x^2} - 2f_xf_y\frac{\partial^2\theta}{\partial x\partial y} + (1 + f_x^2)\frac{\partial^2\theta}{\partial y^2}\right] -$$

$$\left[(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy}\right]\left[f_x\frac{\partial\theta}{\partial x} + f_y\frac{\partial\theta}{\partial y}\right] + \frac{f_x(1+f_x^2+f_y^2)^{1/2}}{(1+f_y^2)} \times$$

$$[(1 + f_y^2)f_{xxy} - 2f_xf_yf_{xyy} + (1 + f_x^2)f_{yyy}]$$

$$+ (1 + f_x^2 + f_y^2)^{-1/2}\left[\begin{array}{c} -4f_xf_yf_{xy}^2 - 2\dfrac{f_xf_y(1 + f_x^2)}{(1 + f_y^2)^2}(2 + f_x^2 + 2f_y^2) \times \\ f_{yy}^2 + (1 - f_x^2 + f_y^2)f_{xx}f_{xy} - 2f_xf_yf_{xx}f_{yy} + \\ \dfrac{1}{(1 + f_y^2)^2}\left\{\begin{array}{l} 1+2f_y^2-f_x^4+f_y^4+9f_x^2f_y^2+3f_x^4f_y^2 \\ + 9f_x^2f_y^4 \end{array}\right\} \\ \times f_{xy}f_{yy} \end{array}\right]$$

$$= 0, \quad (14)$$

where $\theta = \text{arc cot } [p(1 + f_y^2) + f_xf_y]/[1 + f_x^2 + f_y^2]^{1/2}$.

The preceding equation is the necessary and sufficient condition that the ∞^1 curves defined by $dy/dx = p(x, y)$ be an isothermal family on the surface Σ which is given by the Monge equation $z = f(x, y)$.

8. The Condition That the Level Curves z = Const. Form an Isothermal Family.—As an application, we find that the condition that the level curves z = const. of the surface Σ defined by the Monge equation $z = f(x, y)$ be an isothermal family is given by the equation (14) where $\theta = -\text{arc tan } f_y(1 + f_x^2 + f_y^2)^{1/2}/f_x$. Upon simplifying this, we find that the required third order condition is

$$(1 + f_x^2 + f_y^2)(f_x^2 + f_y^2)[f_y(1 + f_y^2)f_{xxx} - f_x(1 + 3f_y^2)f_{xxy} + f_y(1 + 3f_x^2)f_{xyy} - f_x(1 + f_x^2)f_{yyy}] - 2(1 + 2f_x^2 + 2f_y^2)[(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy}] \times [f_xf_yf_{xx} - (f_x^2 - f_y^2)f_{xy} - f_xf_yf_{yy}] = 0. \quad (15)$$

A first integral of this equation is

$$\frac{[(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy}]}{(f_x^2 + f_y^2)(1 + f_x^2 + f_y^2)} = \phi(f). \quad (16)$$

Special classes of solutions are surfaces of revolution with axes perpendicular to the xy-plane, and cylinders with elements parallel to the xy-plane. All minimal surfaces are also solutions as may be verified by (16) since for a minimal surface the numerator of the fraction vanishes. We have proved that there are no surfaces such that the isothermal system of level curves on Σ may be represented on the xy-plane by similar ellipses or hyperbolas with the same axes, or congruent parabolas with the same axis. Hence the only quadric surfaces which belong to our class are those of revolution and the cylinders.

This may be applied to the mapping upon a plane π of the loxodromes (isogonals of the level curves) of the surface Σ. By a theorem of Kasner which states that the complete system of ∞^2 isogonal trajectories of a given family is linear (in the analytic sense) if and only if the given family is isothermal, it can be shown that the loxodromes may be represented by straight lines in the plane π for a sphere (Mercator) and a spheroid (Lambert), but not for an ellipsoid of three unequal axes. Also the loxodromes can be represented by straight lines in π for any minimal surface (in any orientation) and for any surface of revolution with axis perpendicular to π.

In conclusion, we may state that the problem of this section is equivalent to the determination of the class of surfaces Σ which can be projected orthogonally upon a plane π such that the unique Tissot net (the level curves together with their orthogonal trajectories) is isothermal. This suggests for consideration the more general problem of determining the class of surfaces Σ which can be pictured by a given non-conformal transformation T upon a given surface Σ_0 such that the unique orthogonal Tissot net determined by T on Σ or Σ_0 is isothermal. A variation of this problem is to have Σ and Σ_0 given, and then to determine all transformations T with the above property.

Other generalizations and applications of our fundamental formulas will be discussed elsewhere. We show that the only surfaces which are intersected by every set of parallel planes in an isothermal family, besides the obvious cases of spheres and and planes, are the *minimal surfaces*.

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[4] Kasner and De Cicco, "Generalized Transformation Theory of Isothermal and Dual Families," *Proc. Nat. Acad. Sci.*, **28**, 52–55 (1942).

[5] Kasner and De Cicco, "Transformation Theory of Isogonal Trajectories of Isothermal Families," *Ibid.*, **28**, 328–333 (1942).

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NEW TYPES OF RELATIONS IN FINITE FIELD THEORY

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In two other recent papers[1] we developed methods which led to several new results in finite field theory, with particular application to ordinary congruences involving rational integers. In the present article we pursue these methods much further, obtaining various new kinds of relations.

The results just referred to gave criteria involving binomial coefficients for the number of roots of an equation in a finite field. To reduce these expressions to simpler forms in order to give more convenient criteria as to the number of roots, it seems necessary to go into considerations involving binomial coefficients which have not been heretofore studied, and so far we can only use these expressions to supply this information in comparatively few cases, some of which will be discussed elsewhere. At present we shall look at the situation from another angle. It is possible to apply the criteria to certain equations where we know in advance the number of roots or some properties of them. When this is done it turns out to be a fruitful method for finding relations of an entirely new type in number theory.

As one example of this, we obtain immediately from the statement of Theorem II of the first paper[1] and the remarks just preceding it, the result that the least residue, positive or zero, of